A METHOD FOR THE DETERMINATION OF THE EIGENVALUES AND EIGENFUNCTIONS OF A CERTAIN CLASS OF LINEAR INTEGRAL EQUATIONS

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PMM Vol.23, No.3, 1959, pp. 527-533

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(Received 17 July 1958)

This article presents a method for the determination of the eigenvalues and eigenfunctions of Fredholm's linear integral equation with a positive definite root, which is the correlation function of a function of a random variable, and is related to the phenomenon of white noise by a linear differential equation. This method is a modification of the method given in article [1] for the solution of an integral equation of the first kind, occurring in the statistical theory of optimal systems.

1. A short exposition of the method of solution of an integral equation of the first kind. In article [1] a method for the solution of an integral equation of the first kind with finite limits was given. This equation is always reducible to the form

$$\int_0^T K(t, u)\, g(t)\, dt = f(u) \qquad (0 \leqslant u \leqslant T) \tag{1.1}$$

in the case when the real positive definite symmetrical root $K(t, u)$ is the correlation function of a random function $X(t)$, and is related to the white noise $V(t)$ by the linear differential equation

$$FX = HV \qquad \left(D = \frac{d}{dt}\right) \tag{1.2}$$

where F and H are polynomials relative to the operator D, with arbitrary real variable coefficients

$$F = \sum_{k=0}^{n} a_k D^k, \qquad H = \sum_{k=0}^{m} b_k D^k \qquad (m < n) \tag{1.3}$$

In this case, the root of equation (1.1) is expressed by the formula [2, 3]

$$K(t,u)=\int_{-\infty}^{\infty} w(t,\tau)\,w(u,\tau)\,d\tau \tag{1.4}$$

where $\omega(t,\tau)$ is an integral of the differential equation

$$F_t w(t,\tau)=H_t\,\delta(t-\tau) \tag{1.5}$$

equal to zero for $\tau > t$.

The solution of equation (1.1) obtained in article [1] is expressed by the formula

$$g(t)=F^*\eta(t) \tag{1.6}$$

where the function $\eta(t)$ and another function $\xi(t)$ are defined in the interval $0 < t < T$, using the differential equations

$$H^*\eta(t)=\xi(t) \tag{1.7}$$

$$H\xi(t)=Ff(t) \tag{1.8}$$

For negative t the function $\eta(t)$ is defined using the differential equation

$$F^*\eta(t)=0 \qquad (t<0) \tag{1.9}$$

and the functions $\xi(t)$ and $f(t)$ using formula (1.7) and differential equation (1.8) respectively. For all values $t < T$ the functions ξ and η are integrals of equations (1.8) and (1.7), and are defined by the formulas

$$\xi(t)=\int_{-\infty}^{t} w^-(t,\tau)\,f(\tau)\,d\tau, \qquad \eta(t)=\int_{t}^{T} p(\lambda,t)\,\xi(\lambda)\,d\lambda \tag{1.10}$$

where $\omega^-(t,\tau)$ and $p(\lambda, t)$ are the corresponding weighting functions, [1,2]. Formulas (1.10) give the necessary conditions for the determination of all integration constants, after which the functions ξ and η and the solution (1.6) of equation (1.1) are fully determined.

2. An application of the method to the finding of eigenvalues and eigenfunctions. The method presented can be applied after an appropriate modification to the finding of the eigenvalues and eigenfunctions of the homogeneous linear integral equation

$$\int_0^T K(u,t)\,\varphi(t)\,\rho(t)\,dt=\lambda\varphi(u) \qquad (0\leqslant u\leqslant T) \tag{2.1}$$

whose root is determined using formula (1.4) and equation (1.5). Given any non-negative function $\rho(t)$, the equation (2.1) differs formally from the equation (1.2) only in that its right member is the product of an unknown eigenvalue and an unknown function. Therefore, letting in the formulas of the preceding section

$$g(t) = \varphi(t)\rho(t), \qquad f(t) = \lambda\varphi(t) \qquad (0 \leqslant t \leqslant T) \tag{2.2}$$

and making the corresponding changes, we obtain an algorithm for the determination of the eigenvalues and eigenfunctions.

On the basis of the equalities (2.2), formula (1.6) and equation (1.8) can be rewritten in the form

$$\varphi(t) = \frac{1}{\rho(t)} F^* \eta(t) \tag{2.3}$$

$$H\xi(t) = \lambda F \varphi(t) \tag{2.4}$$

Substituting into equation (2.4) the expression (2.3) of the function ϕ, and the expression (1.7) of the function ξ, we shall obtain the following equation determining the function η in the interval $0 < t < T$:

$$F\left(\frac{1}{\rho} F^* \eta\right) - \frac{1}{\lambda} HH^* \eta = 0 \qquad (0 < t < T) \tag{2.5}$$

This is a linear differential equation of order $2n$, containing an unknown parameter λ. Let us denote by $v_1(t, \lambda), \ldots, v_{2n}(t, \lambda)$ any $2n$ linearly independent integrals of equation (2.5). Then the general integral of equation (2.5) will be expressed by the formula

$$\eta(t) = \sum_{\nu=1}^{2n} \gamma_\nu v_\nu(t, \lambda) \qquad (0 < t < T) \tag{2.6}$$

Substituting this expression into (2.3) we shall obtain

$$\varphi(t) = \sum_{\nu=1}^{2n} \gamma_\nu \omega_\nu(t, \lambda) \qquad (0 < t < T) \tag{2.7}$$

where

$$\omega_\nu(t, \lambda) = \frac{1}{\rho(t)} F_t^* v_\nu(t, \lambda) \qquad (\nu = 1, \ldots, 2n) \tag{2.8}$$

In order to determine the unknown integration constants $\gamma_1, \ldots, \gamma_{2n}$ in formulas (2.6) and (2.7), the functions η and f (when $t < 0$) must be determined by an integration of equations (1.9) and (1.8), and then subjected to the corresponding conditions at the end points of the interval $0 < t < T$, which follow from the formulas (1.10). The general integral of equation (1.9) and the corresponding expression for the function f are defined, for $t < 0$, using the formulas [1]

$$\eta(t) = \sum_{r=1}^{n} c_r \eta_r(t), \qquad f(t) = \sum_{r=1}^{n} c_r f_r(t) \qquad (t < 0) \tag{2.9}$$

where $\eta_1, \ldots, \eta_n$ are any linearly independent integrals of equation (1.9) and

$$f_r(t) = \int_{-\infty}^{t} w(t, \tau) H^* \eta_r(\tau)\, d\tau \qquad (r = 1, \ldots, n) \tag{2.10}$$

For the determination of the integration constants $\gamma_1, \ldots, \gamma_{2n}$, $c_1, \ldots, c_n$ we have the following conditions.

First, a solution of the integral equation (2.1) cannot contain δ-functions. Formula (2.3) shows that in order that this be true, it is necessary that the functions $\eta, \eta', \ldots, \eta^{(n-1)}$ be continuous at the points $t = 0$ and $t = T$. This condition gives the $2n$ equations:

$$\sum_{\nu=1}^{2n} \gamma_\nu v_\nu^{(k)}(0, \lambda) - \sum_{r=1}^{n} c_r \eta_r^{(k)}(0) = 0 \qquad (k = 0, 1, \ldots, n-1) \tag{2.11}$$

$$\sum_{\nu=1}^{2n} \gamma_\nu v_\nu^{(k)}(T, \lambda) = 0 \qquad (k = 0, 1, \ldots, n-1) \tag{2.12}$$

Secondly, the function $f(t)$ defined by the second formula of (2.2) for $0 < t < T$, and by the second formula of (2.9) for $t < 0$, must satisfy the equation (1.8) at the point $t = 0$. This condition, together with equation (1.7), expresses the discontinuities of the functions $f, f', \ldots, f^{(n-1)}$ at the point $t = 0$, in terms of the discontinuities of the derivatives of the function η, i.e. it gives n equations. We shall, however, derive these equations in a somewhat different manner. We shall replace the equation (1.8), which is considered as a differential equation with an unknown function $f(t)$, by an equivalent system of equations of the first order, defining the new unknown functions $z_1, \ldots, z_n$ by the formulas

$$\begin{aligned} z_1 &= f, \qquad z_{k+1} = f^{(k)} \qquad (k = 1, \ldots, n-m-1) \\ z_{k+1} &= f^{(k)} - \sum_{h=0}^{k-n+m} (q_k \quad \xi)^{(h)} \qquad (k = n-m, \ldots, n-1) \end{aligned} \tag{2.13}$$

where

$$\begin{aligned} q_{n-m} &= \frac{b_m}{a_n} \qquad (k = n-m+1, \ldots, n) \\ q_k &= \frac{1}{a_n}\left[b_{n-k} - \sum_{h=n-m}^{k-1} \sum_{=0}^{h} C_{n+l-k}^{n-k}\, a_{n+l+h-k}\, q_h^{(L)}\right] \end{aligned} \tag{2.14}$$

Then equation (1.8) will be replaced by the system of equations

$$z_k' = z_{k+1} \quad (k = 1, \ldots, n-m-1), \qquad z_k' = z_{k+1} + q_k\xi \quad (k = n-m, \ldots, n-1)$$
$$z'_n = -\sum_{l=1}^{n} \frac{a_{l-1}}{a_n} z_l + q_n\xi \tag{2.15}$$

By virtue of condition (2.11) and formula (1.7), the function ξ is continuous at the point $t = 0$. Consequently, the integral of the system (2.15) is also continuous at the point $t = 0$. This condition, on the basis of the formulas (2.13), (2.2), (2.7) and (2.9), gives the following n equations:

$$\lambda \sum_{\nu=1}^{2n} \gamma_\nu \omega_\nu^{(k)}(0, \lambda) - \sum_{r=1}^{n} c_r f_r^{(k)}(0) = 0 \qquad (k = 0, 1, \ldots, n-m-1)$$
$$\lambda \sum_{\nu=1}^{2n} \gamma_\nu \left[\omega_\nu^{(k)}(0, \lambda) - \sum_{h=0}^{k-n+m} \sum_{l=0}^{h} C_h^l q_{k-h}^{(h-l)}(0) w_\nu^{(l)}(0, \lambda)\right] -$$
$$- \sum_{r=1}^{n} c_r \left[f_r^{(k)}(0) - \sum_{h=0}^{k-n+m} \sum_{l=0}^{h} C_h^l q_{k-h}^{(h-l)}(0) \xi_r^{(l)}(0)\right] = 0 \tag{2.16}$$
$$(k = n-m, \ldots, n-1)$$

where

$$w_\nu(t, \lambda) = H_t^* v_\nu(t, \lambda) \quad (\nu = 1, \ldots, 2n), \qquad \xi_r(t) = H_t^* \eta_r(t) \quad (r = 1, \ldots, n)$$

The equations (2.11), (2.12) and (2.16) are a system of $3n$ homogeneous linear algebraic equations in $\gamma_1, \ldots, \gamma_{2n}, c_1, \ldots, c_n$, whose coefficients depend on the unknown eigenvalue λ. In order that a non-zero solution of this system exists, it is necessary that its determinant $\Delta(\lambda)$ equals zero

$$\Delta(\lambda) = 0 \tag{2.17}$$

This equation determines the eigenvalues. After the determination of the eigenvalues, equations (2.11), (2.12) and (2.16) will make it possible to express the magnitudes $\gamma_1, \ldots, \gamma_{2n}, c_1, \ldots, c_n$, corresponding to every eigenvalue λ, in terms of one of these. The latter will be determined by the normalizing condition

$$\int_0^T |\varphi(t)|^2 \rho(t)\, dt = 1 \tag{2.18}$$

If for some eigenvalue λ the rank of the coefficient matrix of the system of equations (2.11), (2.12) and (2.16) be less than $2n-1$, and equal, say, to $2n-s$, then this will mean that the given eigenvalue λ is of multiplicity s. In this case, equations (2.11), (2.12) and (2.16) will express all coefficients γ_ν, c_r in terms of any s of them, which can be chosen arbitrarily. This condition can be used for the determination of s systems of values of these s coefficients, such that s orthonormal eigenfunctions are obtained.

The method presented is easily extended to the case of a complex symmetrical root (Hermitian).

In the case when all coefficients of the operators F and H are constant and $\rho(t) \equiv 1$, equation (2.5) is a linear differential equation with constant coefficients. In this case the analytic expressions of the functions η_r, ν_0 and ω_ν are easily found, and the entire problem is solved analytically. Thus, in the case when the root of equation (2.1) is the correlation function of a stationary random function with fractional rational spectral density, the demonstrated method gives a complete analytic solution of the problem of the determination of eigenvalues and eigenfunctions. This solution was obtained earlier by another method in articles [5] and [6].

Example 1. To find the eigenvalues and eigenfunctions of the equation (2.1) in the case when F and H are defined by the formulas

$$F = a_1(t)D + a_0(t), \qquad H = 1 \tag{2.19}$$

and $\rho(t) \equiv 1$. In this case equation (1.5), determining the function $\omega(t, \tau)$ is an equation of the first order which is easily integrated, after which the formula (1.4) gives

$$K(t, t') = \begin{cases} q_1(t)\, q_2(t') & \text{for } t > t' \\ q_1(t')\, q_2(t) & \text{for } t < t' \end{cases} \tag{2.20}$$

where

$$q_1(t) = \exp\left\{-\int_0^t \frac{a_0(\tau)}{a_1(\tau)}\, d\tau\right\}, \qquad q_2(t) = q_1(t) \int_{-\infty}^t \frac{d\tau}{a_1^2(\tau)\, q_1^2(\tau)} \tag{2.21}$$

The equation (2.5) has, in this case, the form

$$(a_1^2\eta')' - \left(a_0^2 + a_0'a_1 - a_0a_1' - a_1a_1'' - \frac{1}{\lambda}\right)\eta = 0 \tag{2.22}$$

Let us denote by $v_1(t, \lambda)$, $v_2(t, \lambda)$ some two linearly independent integrals of equation (2.2). Then formula (2.8) gives

$$\omega_\nu(t, \lambda) = -\frac{d}{dt}\left[a_1(t)\, v_\nu(t, \lambda)\right] + a_0(t) v_\nu(t, \lambda) \qquad (\nu = 1, 2) \tag{2.23}$$

The formula (2.7) which expresses the unknown eigenfunctions will assume the form:

$$\varphi(t) = \gamma_1\omega_1(t, \lambda) + \gamma_2\omega_2(t, \lambda) \tag{2.24}$$

For $t < 0$, the functions η and f are expressed, as in example 1 of article [1], by the formulas

$$\eta(t) = \frac{c_1}{a_1(t)\, q_1(t)}, \qquad f(t) = c_1 q_2(t) \qquad (t < 0) \tag{2.25}$$

Since in the given case $n = 1$, $m = 0$, we have for the determination of the unknown constants γ_1, γ_2 and c_1 one equation (2.16), one equation (2.11), and one equation (2.12):

$$\begin{gathered} \lambda\left[\gamma_1\omega_1(0,\lambda) + \gamma_2\omega_2(0,\lambda)\right] - c_1 q_2(0) = 0 \\ \gamma_1 v_1(0,\lambda) + \gamma_2 v_2(0,\lambda) - \frac{c_1}{a_1(0)\, q_1(0)} = 0 \\ \gamma_1 v_1(T,\lambda) + \gamma_2 v_2(T,\lambda) = 0 \end{gathered} \tag{2.26}$$

Consequently, the equation (2.17), which determines the eigenvalues, will in this case have the form

$$\Delta(\lambda) = \begin{vmatrix} \lambda\omega_1(0,\lambda) & \lambda\omega_2(0,\lambda) & q_2(0) \\ v_1(0,\lambda) & v_2(0,\lambda) & \dfrac{1}{a_1(0)\, q_2(0)} \\ v_1(T,\lambda) & v_2(T,\lambda) & 0 \end{vmatrix} = 0 \tag{2.27}$$

Having determined the eigenvalues by way of solution of the equation (2.27), we shall express two of the unknown coefficients γ_1, γ_2, c_1, in terms of the third. The latter will be determined from the normalizing condition (2.18).

Example 2. A more detailed examination of a particular case of the preceding example, when the coefficients a_0 and a_1 are constant and equal respectively to

$$a_0 = \sqrt{\frac{\alpha}{2}}, \qquad a_1 = \frac{1}{\sqrt{2\alpha}}$$

In this case

$$q_1(t) = \frac{1}{q_2(t)} = e^{-\alpha t} \tag{2.28}$$

and formula (2.20) assumes the form

$$K(t, t') = e^{-\alpha|t-t'|} \tag{2.29}$$

Equation (2.2) has, in this case, the form

$$\eta'' + \left(\frac{2\alpha}{\lambda} - \alpha^2\right)\eta = 0 \tag{2.30}$$

This equation with constant coefficients has two linearly independent integrals of the form

$$v_1(t,\lambda) = e^{i\omega t}, \qquad v_2(t,\lambda) = e^{-i\omega t} \tag{2.31}$$

where

$$\omega = \sqrt{\frac{2\alpha}{\lambda} - \alpha^2} \tag{2.32}$$

The formulas (2.3) give

$$\omega_1(t,\lambda) = \frac{\alpha - i\omega}{2\alpha} e^{i\omega t}, \qquad \omega_2(t,\lambda) = \frac{\alpha + i\omega}{2\alpha} e^{-i\omega t} \tag{2.33}$$

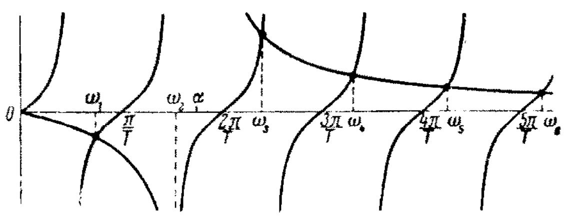
The equation (2.27) assumes the form:

$$\begin{vmatrix} \lambda \dfrac{\alpha - i\omega}{2\alpha} & \lambda \dfrac{\alpha + i\omega}{2\alpha} & 1 \\ 1 & 1 & \sqrt{2\alpha} \\ e^{i\omega T} & e^{-i\omega T} & 0 \end{vmatrix} = 0 \tag{2.34}$$

Expanding the determinant, taking into consideration (2.32), and performing the elementary transformations, we can bring equation (2.34) to the form

$$\tan \omega T = -\frac{2\alpha\omega}{\alpha^2 - \omega^2} \tag{2.35}$$

This equation determines an infinitely increasing sequence of values $\omega = \omega_1, \omega_2, \ldots$. In the figure a graphical solution of the equation (2.35) is represented for the case when $\alpha = 7\pi / 4T$



Substituting the value $\omega = \omega_\nu$, determined by the equation (2.35) in (2.32), and solving the resulting equation with respect to λ, we shall find the corresponding eigenvalue

$$\lambda_\nu = \frac{2\alpha}{\alpha^2 + \omega_\nu^2} \qquad (\nu = 1, 2, \ldots) \tag{2.36}$$

This formula gives an infinitely decreasing sequence of eigenvalues.

The system of equations (2.26) in this case has the form

$$\lambda [(\alpha - i\omega) \gamma_1 + (\alpha + i\omega) \gamma_2] - \sqrt{2\alpha}\, c_1 = 0$$

$$\gamma_1 + \gamma_2 - \sqrt{2\alpha}\, c_1 = 0, \qquad \gamma_1 e^{i\omega T} + \gamma_2 e^{-i\omega T} = 0$$

The last of these equations gives $\gamma_2 = -\gamma_1 e^{2i\omega T}$. Substituting this expression into any of the two remaining equations of (2.37), we can also express c_1 in terms of γ_1. This, however, is not necessary, since the constants c_r are auxiliary, and do not directly enter into the solution of the problem.

Substituting the found expressions (2.33) and γ_2 into formula (2.24), and performing elementary transformations, and then determining γ_1, from the normalizing condition (2.18), we find the eigenfunctions

$$\varphi_\nu (t) = \sqrt{\frac{2}{T + \lambda_\nu}} \sin \left[\omega_\nu \left(t - \frac{T}{2}\right) + \frac{\nu\pi}{2}\right] \qquad (\nu = 1, 2, \ldots) \qquad (2.38)$$

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Translated by I.S.